

Fused Mackey functors

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Abstract: Let G be a finite group. In [5], Hambleton, Taylor and Williams have considered the question of comparing Mackey functors for G and biset functors defined on subgroups of G and bifree bisets as morphisms.

This paper proposes a different approach to this problem, from the point of view of various categories of G -sets. In particular, the category $G\text{-}\underline{\mathbf{set}}$ of *fused G -sets* is introduced, as well as the category $\underline{\mathbf{S}}(G)$ of spans in $G\text{-}\underline{\mathbf{set}}$. The *fused Mackey functors* for G over a commutative ring R are defined as R -linear functors from $R\underline{\mathbf{S}}(G)$ to R -modules. They form an abelian subcategory $\mathbf{Mack}_R^f(G)$ of the category of Mackey functors for G over R . The category $\mathbf{Mack}_{\mathbb{Z}}^f(G)$ is equivalent to the category of conjugation Mackey functors of [5]. The category $\mathbf{Mack}_R^f(G)$ is also equivalent to the category of modules over the *fused Mackey algebra* $\mu_R^f(G)$, which is a quotient of the usual Mackey algebra $\mu_R(G)$ of G over R .

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1. Introduction

This note is devoted to the frequently asked question of comparing Mackey functors for a single finite group G with biset functors defined only on subgroups of G and left-right free bisets as morphisms. The answer to this question has already been given by Hambleton, Taylor and Williams ([5]), but in a rather computational and non canonical way (in particular, in Section 7, the definition of the functor j_{\bullet} requires the choice of sets of representatives of orbits of any finite G -set).

The present paper makes a systematic use of Dress definition ([3]) and Lindner definition ([6]) of Mackey functors, to avoid these non canonical choices. This leads to the definition of the category of *fused G -sets* (Section 3), and the category of *fused Mackey functors* (Section 4) for a finite group G , which is equivalent to the category of “conjugation invariant Mackey functors” of [5]. This category is also equivalent to the category of modules over the *fused Mackey algebra*, introduced in Section 5.

2. Conjugation bisets revisited

2.1. First a notation : when G is a finite group, and X is a finite G -set, let $G\text{-}\underline{\mathbf{set}}\downarrow_X$ denote the category of (finite) G -sets over X : its objects are pairs (Y, b) consisting of a finite G -set Y , and a morphism of G -sets $b : Y \rightarrow X$. A

morphism $f : (Y, b) \rightarrow (Z, c)$ in $G\text{-}\mathbf{set}\downarrow_X$ is a morphism of G -sets $f : Y \rightarrow Z$ such that $c \circ f = b$.

There is an obvious notion of disjoint union in $G\text{-}\mathbf{set}\downarrow_X$, and the corresponding Grothendieck group is called the Burnside group over X . It will be denoted by $\mathcal{B}(G X)$, or $\mathcal{B}(X)$ when G is clear from the context.

Similarly, when G and H are finite groups, and U is a (G, H) -biset, one can define the category $(G, H)\text{-}\mathbf{biset}\downarrow_U$ of (G, H) -bisets over U , and the Burnside group $\mathcal{B}(G U_H)$ of (G, H) -bisets over U .

2.2. When H is a subgroup of G , and Y is an H -set, induction from H -sets to G -sets is an equivalence of categories from $H\text{-}\mathbf{set}\downarrow_Y$ to $G\text{-}\mathbf{set}\downarrow_{\text{Ind}_H^G Y}$. A quasi-inverse equivalence is the functor sending the G -set (X, a) over $\text{Ind}_H^G Y$ to the H -set $a^{-1}(1 \times_H Y)$ (see [2] Lemma 2.4.1). In particular $\mathcal{B}(H Y) \cong \mathcal{B}(G \text{Ind}_H^G Y)$.

2.3. Now an observation: when H and K are subgroups of G , the conjugation (K, H) -bisets defined in Section 6 of [5] are exactly those over the biset ${}_K G_H$ (the set G on which K and H act by multiplication), i.e. the (K, H) -bisets U for which there exists a biset morphism $U \rightarrow {}_K G_H$.

Indeed, a conjugation (K, H) -biset U is a bifree (K, H) -biset isomorphic to a disjoint union of bisets of the form $(K \times H)/S$, where S is a subgroup of $K \times H$ of the form

$$S_{g,A} = \{(g x, x) \mid x \in A\}$$

where A is a subgroup of H , and g is an element of G such that ${}^g A \leq K$. For such a transitive biset $(K \times H)/S$, the map

$$\forall (k, h)S \in (K \times H)/S, (k, h)S \mapsto kgh^{-1}$$

is a morphism of (K, H) -bisets.

Conversely, let U be a (K, H) -biset for which there exists a biset morphism $\alpha : U \rightarrow {}_K G_H$. Then for any $u \in U$, the stabilizer S_u of u in $K \times H$ is the subgroup

$$S_u = \{(k, h) \in K \times H \mid k \cdot u \cdot h^{-1} = u\}$$

of $K \times H$. Then if $(k, h) \in S_u$,

$$\alpha(k \cdot u) = k\alpha(u) = \alpha(u \cdot h) = \alpha(u)h \quad .$$

Let A_u denote the projection of S_u into H , and set $g_u = \alpha(u)$. It follows that $S_u \subseteq S_{g_u, A_u}$.

Conversely, if $(k, h) \in S_{g_u, A_u}$, then $k = {}^{g_u}h$, and there exists some $x \in K$ such that $(x, h) \in S_u$, since $h \in A_u$. Thus $x \cdot u \cdot h^{-1} = u$, from which follows that

$$\alpha(x \cdot u) = xg_u = \alpha(u \cdot h) = g_u h \quad ,$$

hence $x = {}^{g_u}h = k$, and $S_u = S_{g_u, A_u}$. Observation 2.3 follows.

2.4. In other words, conjugation (K, H) -bisets form a category $\mathbf{Conj}_{K, H}^G$, and there is a forgetful functor $\Phi : (K, H)\text{-biset} \downarrow_{K G_H} \rightarrow \mathbf{Conj}_{K, H}^G$ sending (U, a) to U . This functor is full, preserves disjoint unions, and moreover it induces a surjection on the corresponding sets of isomorphism classes. This means that Φ induces a surjective group homomorphism (still denoted by Φ) from $\mathcal{B}(K G_H)$ to the Grothendieck group $\mathcal{B}_{K, H}^G$ of conjugation (K, H) -bisets.

2.5. If H, K and L are subgroups of G , if (U, a) is a (K, H) -biset over ${}_K G_H$ and (V, b) is an (L, K) -biset over ${}_L G_K$, the composition $(V, b) \circ (U, a)$ is the (L, H) -biset over ${}_L G_H$ defined by the following diagram:

$$\begin{array}{ccc} \begin{array}{c} V \\ \downarrow b \\ {}_L G_K \end{array} & \circ & \begin{array}{c} U \\ \downarrow a \\ {}_K G_H \end{array} = \begin{array}{c} V \times_K U \\ \downarrow b \times_K a \\ G \times_K G \\ \downarrow \mu \\ {}_L G_H \end{array} \end{array}$$

where μ is multiplication in G . This composition is associative, and additive with respect to disjoint unions. Hence it induces a composition

$$\widehat{\circ} : \mathcal{B}({}_L G_K) \times \mathcal{B}({}_K G_H) \rightarrow \mathcal{B}({}_L G_H) \quad .$$

Hence, one can define a category $\widehat{\mathbf{B}}(G)$ whose objects are the subgroups of G , and such that $\text{Hom}_{\widehat{\mathbf{B}}(G)}(H, K) = \mathcal{B}({}_K G_H)$, for subgroups H and K of G . Composition is given by $\widehat{\circ}$, and the identity morphism of the subgroup H of G in the category $\widehat{\mathbf{B}}(G)$ is the class of the biset $({}_H H_H, i_H)$, where $i_H : {}_H H_H \rightarrow {}_H G_H$ is the inclusion map from H to G .

Since the functor Φ maps the composition $\widehat{\circ}$ to the composition of bisets, and the identity morphism of H in $\widehat{\mathbf{B}}(G)$ to the identity biset ${}_H H_H$, one can extend Φ to a functor $\widehat{\mathbf{B}}(G) \rightarrow \mathbf{B}(G)$, which is the identity on objects.

In other words, the category $\mathbf{B}(G)$ introduced in Section 3 of [5] is the quotient of the category $\widehat{\mathbf{B}}(G)$ obtained by identifying morphisms which have the same image by Φ .

2.6. By the above Remark 2.2, when H and K are subgroups of G , there is

a group isomorphism

$$\mathcal{B}({}_K G_H) \cong \mathcal{B}(\text{Ind}_{K \times H}^{G \times G}({}_K G_H)) \quad ,$$

(with the usual identification of (K, H) -bisets with $(K \times H)$ -sets). Now the biset ${}_K G_H$ is actually the restriction to $(K \times H)$ of the (G, G) -biset G . By the Frobenius reciprocity, it follows that

$$\text{Ind}_{K \times H}^{G \times G}({}_K G_H) \cong \text{Ind}_{K \times H}^{G \times G} \text{Res}_{K \times H}^{G \times G}(G G_G) \cong (\text{Ind}_{K \times H}^{G \times G} \bullet) \times_G G_G \quad ,$$

where \bullet is a set of cardinality 1. Since $\text{Ind}_{K \times H}^{G \times G} \bullet \cong (G/K) \times (G/H)$, it follows (after switching G/H and G) that

$$\text{Ind}_{K \times H}^{G \times G}({}_K G_H) \cong (G/K) \times G \times (G/H) \quad ,$$

where the (G, G) -biset structure of the right hand side is given by

$$\forall (a, b, x, y, g) \in G^5, \quad a \cdot (xK, g, yH) \cdot b = (axK, agb, b^{-1}yH) \quad .$$

2.7. It should now be clear that the additive completion $\widehat{\mathbf{B}}_{\bullet}(G)$ is equivalent to the category whose objects are finite G -sets, where for any two finite G -sets X and Y

$$\text{Hom}_{\mathbf{B}_{\bullet}(G)}(X, Y) = \mathcal{B}({}_G(Y \times G \times X)_G) \quad ,$$

the (G, G) -biset structure on $(Y \times G \times X)$ being given as above by

$$\forall (a, b, g, x, y) \in G^3 \times X \times Y, \quad a \cdot (y, g, x) \cdot b = (ay, agb, b^{-1}x) \quad .$$

Keeping track of the composition $\widehat{\circ}$ along the above isomorphism shows that the composition in the category $\widehat{\mathbf{B}}_{\bullet}(G)$ can be defined by linearity from the following: if X, Y , and Z are finite G -sets, if

$$\begin{array}{ccc} & V & \\ f \swarrow & \downarrow e & \searrow d \\ Z & G & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} & U & \\ c \swarrow & \downarrow b & \searrow a \\ Y & G & X \end{array}$$

are (G, G) -bisets over $(Z \times G \times Y)$ and $(Y \times G \times X)$, respectively, their composition is given by the following (G, G) -biset over $(Z \times G \times X)$

$$\begin{array}{ccc} & (V \times_{d,c} U)/G & \\ \gamma \swarrow & \downarrow \beta & \searrow \alpha \\ Z & G & X \end{array}$$

where $V \times_{d,c} U$ is the pullback of V and U over Y , i.e. the set of pairs $(v, u) \in V \times U$ with $d(v) = c(u)$, and $(V \times_{d,c} U)/G$ the set of orbits of G on it for the action given by $(v, u) \cdot g = (vg, g^{-1}u)$. This makes sense because $d(v \cdot g) = g^{-1}d(v) = g^{-1}c(u) = c(g^{-1} \cdot u)$ if $d(v) = c(u)$. The map (γ, β, α) is given by

$$(\gamma, \beta, \alpha)((v, u)G) = (f(v), e(v)b(u), a(u)) \quad .$$

2.8. The functor $\Phi : \widehat{\mathbf{B}}(G) \rightarrow \mathbf{B}(G)$ extends uniquely to an additive functor $\Phi_{\bullet} : \widehat{\mathbf{B}}_{\bullet}(G) \rightarrow \mathbf{B}_{\bullet}(G)$, and the category $\mathbf{B}_{\bullet}(G)$ is the quotient of $\widehat{\mathbf{B}}_{\bullet}(G)$ obtained by identifying morphisms which have the same image by Φ_{\bullet} . Clearly, two morphisms $f, g \in \text{Hom}_{\widehat{\mathbf{B}}_{\bullet}(G)}(X, Y)$ are identified if and only if $f - g$ is in the kernel of the group homomorphism

$$\phi : \mathcal{B}_{(G(Y \times G \times X)_G)} \rightarrow \mathcal{B}_{(G(Y \times X)_G)}$$

induced by the correspondence

$$\begin{array}{ccc} & U & \\ c \swarrow & \downarrow b & \searrow a \\ Y & G & X \end{array} \quad \mapsto \quad \begin{array}{ccc} & U & \\ c \swarrow & & \searrow a \\ Y & & X \end{array}$$

on bisets. In other words, a morphism f in $\widehat{\mathbf{B}}_{\bullet}(G)$ gives the zero morphism in $\mathbf{B}_{\bullet}(G)$ if and only if it belongs to $\text{Ker } \phi$.

2.9. Now the (G, G) -biset ${}_G G_G$ is isomorphic to $\text{Ind}_{\Delta(G)}^{G \times G} \bullet$, where $\Delta(G)$ is the diagonal subgroup of $G \times G$. It follows that there is an isomorphism of (G, G) -bisets

$$Y \times G \times X \cong \text{Ind}_{\Delta(G)}^{G \times G}(Y \times X) \quad .$$

Hence, by Remark 2.2 again, since $\Delta(G) \cong G$,

$$\mathcal{B}_{(G(Y \times G \times X)_G)} \cong \mathcal{B}_{(G(Y \times X))} \quad ,$$

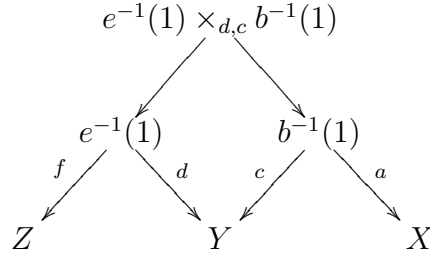
where ${}_G(Y \times X)$ is the usual cartesian product with diagonal G -action. More precisely, this isomorphism is induced by the correspondence

$$\begin{array}{ccc} & U & \\ c \swarrow & \downarrow b & \searrow a \\ Y & G & X \end{array} \quad \mapsto \quad \begin{array}{ccc} & b^{-1}(1) & \\ c \swarrow & & \searrow a \\ Y & & X \end{array}$$

It is then easy to check that the composition of



corresponds to the usual pullback diagram



In other words, the category $\widehat{\mathbf{B}}_{\bullet}(G)$ is equivalent to the category $\mathbf{S}(G)$ whose objects are the finite G -sets, where

$$\mathrm{Hom}_{\mathbf{S}(G)}(X, Y) = \mathcal{B}(G(Y \times X)) \quad ,$$

and composition is induced by pullback. It has been shown by Lindner ([6], see also [2]) that the additive functors on this category are precisely the Mackey functors for G .

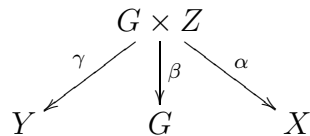
2.10. It remains to keep track of identifications by Φ , i.e. to start with a morphism $f \in \mathrm{Hom}_{\mathbf{S}(G)}(X, Y)$, to lift it to

$$f^+ \in \mathrm{Hom}_{\widehat{\mathbf{B}}_{\bullet}(G)}(X, Y) = \mathcal{B}(G(Y \times G \times X)_G) \quad ,$$

and see when f^+ lies in $\mathrm{Ker} \phi$. Now f is represented by a difference of two G -sets over $G(Y \times X)$ of the form



By induction from $\Delta(G)$ to $G \times G$, the G -set on the left hand side lifts to the following $(G \times G)$ -set over $(G \times G)(Y \times G \times X)$



where the $(G \times G)$ -actions on $G \times Z$ and $Y \times G \times X$ are given respectively by $(s, t) \cdot (g, z) = (sgt^{-1}, tz)$ and $(s, t) \cdot (y, g, x) = (sy, sgt^{-1}, tx)$, and where

$$(\gamma, \beta, \alpha)(g, z) = (gb(z), g, a(z)) \quad .$$

Similarly the G -set $(Z', (b', a'))$ lifts to $(G \times Z', (\gamma', \beta', \alpha'))$.

Now f^+ is in $\text{Ker } \phi$ if and only if there is an isomorphism

$$\begin{array}{ccc} & G \times Z & \\ \gamma \swarrow & & \searrow \alpha \\ Y & & X \end{array} \quad \xrightarrow{\theta} \quad \begin{array}{ccc} & G \times Z' & \\ \gamma' \swarrow & & \searrow \alpha' \\ Y & & X \end{array} \quad .$$

of $(G \times G)$ -sets over $Y \times X$. Since $(g, z) = g \cdot (1, z)$ for any $(g, z) \in G \times Z$, it follows that θ is a map from $G \times Z$ to $G \times Z'$ of the form

$$(g, z) \mapsto (gu(z), v(z)) \quad ,$$

where u is a map from Z to G and v is a map from Z to Z' . Now for any $(s, t) \in G \times G$, the equality

$$\theta((s, t) \cdot (g, z)) = (s, t) \cdot \theta((g, z))$$

gives

$$(sgt^{-1}u(tz), v(tz)) = (sgu(z)t^{-1}, tv(z)) \quad .$$

This is equivalent to

$$u(tz) = {}^t u(z) \quad \text{and} \quad v(tz) = tv(z) \quad .$$

This means that u is a morphism of G -sets from Z to G^c , which is the set G with G -action by conjugation, and v is a morphism of G -sets.

Moreover θ is a bijection if and only if v is.

Finally θ is an morphism of (G, G) -bisets *over* $Y \times X$ if and only if $\alpha' \circ \theta = a$ and $\gamma' \circ \theta = \gamma$, i.e. equivalently if

$$a' \circ v = a \quad \text{and} \quad gu(z) \cdot b' \circ v(z) = g \cdot b(z)$$

for any $(g, z) \in G \times Z$. In other words

$$a = a' \circ v \quad \text{and} \quad b = u * (b' \circ v) \quad ,$$

where, for any map $w : Z \rightarrow Y$, the map $u * w : Z \rightarrow Y$ is defined by $(u * w)(z) = u(z) \cdot w(z)$. The map $u * w$ is a map of G -sets if $u : Z \rightarrow G^c$

and $w : Z \rightarrow Y$ are. Note that $w' = u * w$ if and only if $w = \bar{u} * w'$, where $\bar{u} : Z \rightarrow G^c$ is defined by $\bar{u}(z) = u(z)^{-1}$.

It follows that f maps to the zero morphism in $\mathbf{B}(G)$ if and only if there exists $u : Z \rightarrow G^c$ and an isomorphism $v : Z \rightarrow Z'$ such that

$$a' \circ v = a \quad \text{and} \quad b' \circ v = u * b \quad ,$$

But then v is an isomorphism

$$\begin{array}{ccc} & Z & \\ b' \circ v \swarrow & & \searrow a' \circ v \\ Y & & X \end{array} \xrightarrow{v} \begin{array}{ccc} & Z' & \\ b' \swarrow & & \searrow a' \\ Y & & X \end{array} .$$

of G -sets over $Y \times X$, and f is also represented by the difference

$$\begin{array}{ccc} & Z & \\ b \swarrow & & \searrow a \\ Y & & X \end{array} - \begin{array}{ccc} & Z & \\ u*b \swarrow & & \searrow a \\ Y & & X \end{array} ,$$

since $a' \circ v = a$ and $b' \circ v = u * b$. These are the morphisms in the category $\mathbf{S}(G)$ that vanish in $\mathbf{B}_\bullet(G)$. In other words:

2.11. Theorem : *Let G be a finite group. Let $\underline{\mathbf{S}}(G)$ denote the quotient category of $\mathbf{S}(G)$ defined by setting, for any two finite G -sets Y and X*

$$\mathrm{Hom}_{\underline{\mathbf{S}}(G)}(X, Y) = \mathcal{B}_G(Y \times X) / K(Y, X) \quad ,$$

where $K(Y, X)$ is the subgroup generated by the differences

$$(2.12) \quad \begin{array}{ccc} & Z & \\ b \swarrow & & \searrow a \\ Y & & X \end{array} - \begin{array}{ccc} & Z & \\ u*b \swarrow & & \searrow a \\ Y & & X \end{array} ,$$

where $a : Z \rightarrow X$, $b : Z \rightarrow Y$, and $u : Z \rightarrow G^c$ are morphisms of G -sets.

Then the functor Φ_\bullet induces an equivalence of categories $\underline{\mathbf{S}}(G) \cong \mathbf{B}_\bullet(G)$.

Since the difference 2.12 factors as

$$\begin{array}{c} Z \\ \swarrow \text{Id} \searrow b \\ Y \quad Z \end{array} \circ \left(\begin{array}{c} Z \\ \swarrow \text{Id} \searrow \text{Id} \\ Z \quad Z \end{array} - \begin{array}{c} Z \\ \swarrow u*\text{Id} \searrow \text{Id} \\ Z \quad Z \end{array} \right) \circ \begin{array}{c} Z \\ \swarrow \text{Id} \searrow a \\ Z \quad X \end{array}$$

the morphism vanishing in $\underline{\mathbf{S}}(G)$ are generated in the category $\mathbf{S}(G)$ by the morphisms of the form

$$\begin{array}{c} Z \\ \swarrow \text{Id} \searrow \text{Id} \\ Z \quad Z \end{array} - \begin{array}{c} Z \\ \swarrow u*\text{Id} \searrow \text{Id} \\ Z \quad Z \end{array} .$$

2.13. It follows that the additive functors from $\underline{\mathbf{S}}(G)$ to the category of abelian groups are exactly those Mackey functors (in the sense of Dress) such that for any G -set Z and any $u : Z \rightarrow G^c$, the morphism $M_*(u * \text{Id})$ is equal to the identity map of $M(Z)$.

This condition is additive with respect to Z , since the map $u * \text{Id}_Z$ maps each G -orbit of Z to itself. Hence these functors are exactly the functors for which the map $M_*(u * \text{Id})$ is the identity map of $M(G/H)$, for any subgroup H of G and any $u : G/H \rightarrow G^c$. Such a map is of the form $gH \mapsto gcH$, where $c \in C_G(H)$. The map $u * \text{Id} : G/H \rightarrow G/H$ is the map $gH \mapsto gcH$.

Translated in terms of the usual definition of Mackey functors, this map expresses the action of c on $M(H) = M(G/H)$. This shows that additive functors from $\underline{\mathbf{S}}(G)$ to abelian groups are exactly the Mackey functors for the group G such that, for any $H \leq G$, the centralizer $C_G(H)$ acts trivially on $M(H)$. These are the “conjugation invariant Mackey functors” introduced in [5].

3. Fused G -sets

Let Z be any (finite) G -set. The multiplication $(u, v) \mapsto u * v$ endows the set $\text{Hom}_{G\text{-set}}(Z, G^c)$ with a group structure. Moreover, for any finite G -set X , this group acts on the left on the set $\text{Hom}_{G\text{-set}}(Z, X)$, via $(u, f) \mapsto u * f$. This action is compatible with the composition of morphisms: if Y is a finite G -set, if $u : Z \rightarrow G^c$ and $v : Y \rightarrow G^c$ are morphisms of G -sets, then for any morphisms of G -sets $f : Z \rightarrow Y$ and $g : Y \rightarrow X$, one checks easily that

$$(3.1) \quad (v * g) \circ (u * f) = (u * (v \circ f)) * (g \circ f) .$$

3.2. Notation : Let $G\text{-}\underline{\text{set}}$ denote the category of fused G -sets: its objects are finite G -sets, and for any finite G -sets Z and Y

$$\text{Hom}_{G\text{-}\underline{\text{set}}}(Z, Y) = \text{Hom}_{G\text{-}\text{set}}(Z, G^c) \setminus \text{Hom}_{G\text{-}\text{set}}(Z, Y) \quad .$$

The composition of morphisms in $G\text{-}\underline{\text{set}}$ is induced by the composition of morphisms in $G\text{-}\text{set}$.

3.3. Remark : For any G -set Y , set $Y^I = Y \times G^c$. This notation is chosen to evoke a path object in homotopy theory (cf. [4] Section 4.12). There is a natural morphism $p : Y^I \rightarrow Y \times Y$, defined by $p(y, g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \rightarrow Y^I$ defined by $i(y) = (y, 1)$, for $y \in Y$. The composition $p \circ i$ is equal to the diagonal map $Y \rightarrow Y \times Y$.

Two morphisms $a, b : Z \rightarrow Y$ in $G\text{-}\text{set}$ are equal in the category $G\text{-}\underline{\text{set}}$ if and only if the morphism $(a, b) : Z \rightarrow Y \times Y$ factors as

$$\begin{array}{ccc} & & Y^I \\ & \nearrow \varphi & \downarrow p \\ Z & \xrightarrow{(a,b)} & Y \times Y \end{array}$$

for some morphism of G -sets $\varphi : Z \rightarrow Y^I$.

3.4. Remark : It follows from 3.1 that the map $u \mapsto u * \text{Id}_Z$ is a group antihomomorphism from $\text{Hom}_{G\text{-}\text{set}}(Z, G^c)$ to the group of G -automorphisms of Z . Hence a morphism $\underline{f} : Z \rightarrow Y$ in the category $G\text{-}\underline{\text{set}}$ is an isomorphism if and only if any of its representatives $f : Z \rightarrow Y$ in $G\text{-}\text{set}$ is an isomorphism.

3.5. Weak pullbacks of fused G -sets. Disjoint union of G -sets is a coproduct in $G\text{-}\underline{\text{set}}$. There is also a weak version of pullback in $G\text{-}\underline{\text{set}}$: let

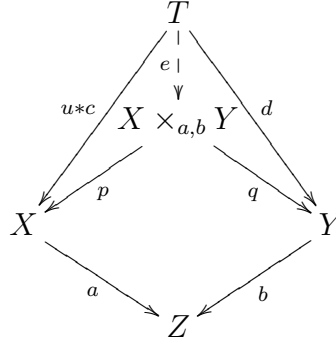
$$\begin{array}{ccc} & T & \\ \underline{c} \swarrow & & \searrow \underline{d} \\ X & & Y \\ \underline{a} \searrow & & \swarrow \underline{b} \\ & Z & \end{array}$$

be a commutative diagram in $G\text{-}\underline{\text{set}}$, where underlines denote the images in $G\text{-}\underline{\text{set}}$ of morphisms in $G\text{-}\text{set}$. This means that $\underline{a} \circ \underline{c} = \underline{b} \circ \underline{d}$, i.e. that there

exists $u \in \text{Hom}_{G\text{-set}}(T, G^c)$ such that

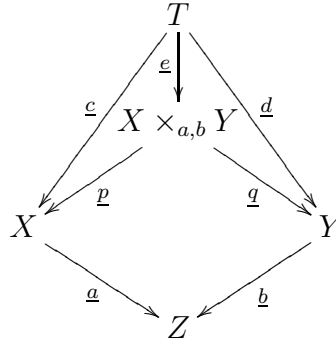
$$b \circ d = u * (a \circ c) \quad .$$

But $u * (a \circ c) = a \circ (u * c)$. It follows that there is a unique morphism $e \in \text{Hom}_{G\text{-set}}(T, X \times_{a,b} Y)$ such that the diagram



is commutative in $G\text{-set}$, where $p : X \times_{a,b} Y \rightarrow X$ and $q : X \times_{a,b} Y \rightarrow Y$ are the canonical morphisms from the pullback $X \times_{a,b} Y$. In other words, the diagram

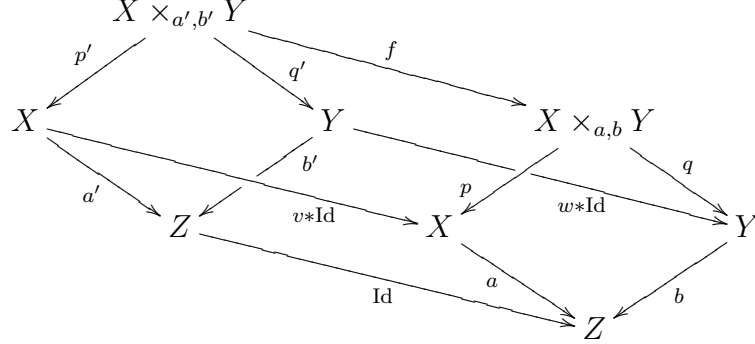
(3.6)



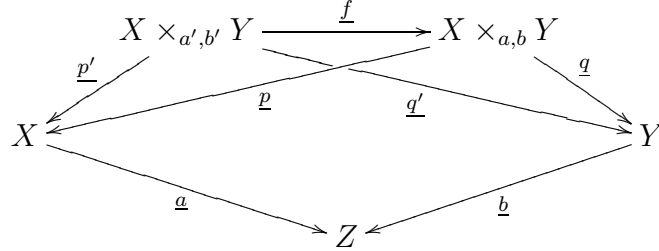
is commutative in $G\text{-set}$.

But still $(X \times_{a,b} Y, \underline{p}, \underline{q})$ need not be a pullback in $G\text{-set}$, since the morphism \underline{e} making Diagram 3.6 commutative is generally not unique, as e itself depends on the choice of u . Moreover, the lifts \underline{a} and \underline{b} of a and b to $G\text{-set}$ are not unique : it should be noted however that if $\underline{a}' = v * \underline{a}$ and $\underline{b}' = w * \underline{b}$ are other lifts of a and b , respectively, where $v \in \text{Hom}_{G\text{-set}}(X, G^c)$ and $w \in \text{Hom}_{G\text{-set}}(Y, G^c)$, then the map $f : (x, y) \mapsto (v(x)x, w(y)y)$ is an

isomorphism of G -sets from $X \times_{a',b'} Y$ to $X \times_{a,b} Y$, such that the diagram

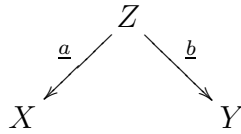


is commutative in $G\text{-}\underline{\text{set}}$. Since $\underline{a}' = \underline{a}$, $\underline{b}' = \underline{b}$, $\underline{v} * \text{Id} = \text{Id}$, and $\underline{w} * \text{Id} = \text{Id}$, this yields a commutative diagram

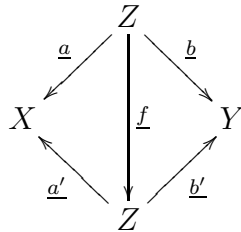


in $G\text{-}\underline{\text{set}}$, and \underline{f} is an isomorphism. This shows that the weak pullback $X \times_{a,b} Y$ only depends on \underline{a} and \underline{b} in the category $G\text{-}\underline{\text{set}}$. For this reason, it may be denoted by $X \times_{\underline{a}, \underline{b}} Y$.

3.7. Spans of fused G -sets. Recall (cf. [9], [1] for the general definition) that if X and Y are finite G -sets, then a *span* $\Lambda_{Z, \underline{a}, \underline{b}}$ over X and Y in the category $G\text{-}\underline{\text{set}}$ is a diagram of the form



where Z is a finite G -set and $\underline{a}, \underline{b}$ are morphisms in the category $G\text{-}\underline{\text{set}}$. Two spans $\Lambda_{Z, \underline{a}, \underline{b}}$ and $\Lambda_{Z', \underline{a}', \underline{b}'}$ over X and Y are equivalent if there exists an isomorphism $\underline{f} : Z \rightarrow Z'$ in $G\text{-}\underline{\text{set}}$ such that the diagram



is commutative. The set of equivalence classes of spans of fused G -sets over X and Y is an additive monoid, where the addition is defined by disjoint union (i.e. $\Lambda_{Z_1, \underline{a}_1, \underline{b}_1} + \Lambda_{Z_2, \underline{a}_2, \underline{b}_2} = \Lambda_{Z_1 \sqcup Z_2, \underline{a}_1 \sqcup \underline{a}_2, \underline{b}_1 \sqcup \underline{b}_2}$). The corresponding Grothendieck group is isomorphic to $\text{Hom}_{\underline{\mathbf{S}}(G)}(Y, X)$.

It should be noted that even if there is no pullback construction in the category $G\text{-}\underline{\mathbf{set}}$, the isomorphism classes of spans in $G\text{-}\underline{\mathbf{set}}$ can still be composed by *weak pullback*, and this induces the composition of morphisms in $\underline{\mathbf{S}}(G)$.

4. Fused Mackey functors

4.1. Definition : Let R be a commutative ring. Let $R\mathbf{S}(G)$ (resp. $R\underline{\mathbf{S}}(G)$) denote the R -linear extension of the category $\mathbf{S}(G)$ (resp. $\underline{\mathbf{S}}(G)$), defined as follows:

- The objects of $R\mathbf{S}(G)$ and $R\underline{\mathbf{S}}(G)$ are finite G -sets.
- For finite G sets X and Y ,

$$\text{Hom}_{R\mathbf{S}(G)}(X, Y) = R \otimes_{\mathbb{Z}} \text{Hom}_{\mathbf{S}(G)}(X, Y) ,$$

$$\text{Hom}_{R\underline{\mathbf{S}}(G)}(X, Y) = R \otimes_{\mathbb{Z}} \text{Hom}_{\underline{\mathbf{S}}(G)}(X, Y) .$$

- Composition of morphisms is induced by the pullback in $G\text{-}\mathbf{set}$ (resp. the weak pullback in $G\text{-}\underline{\mathbf{set}}$).

A Mackey functor for G over R in the sense of Lindner ([6]) is an R -linear functor from $R\mathbf{S}(G)$ to the category $R\text{-Mod}$ of R -modules.

Similarly, a fused Mackey functor for G over R is an R -linear functor from $R\underline{\mathbf{S}}(G)$ to $R\text{-Mod}$. A morphism of fused Mackey functors is a natural transformation of functors. Fused Mackey functors for G over R form a category denoted by $\text{Mack}_R^f(G)$.

The following is an equivalent definition of fused Mackey functors, à la Dress:

4.2. Definition : Let R be a commutative ring. A fused Mackey functor for the group G over R is a bivariant R -linear functor $M = (M^*, M_*)$ from $G\text{-}\underline{\mathbf{set}}$ to $R\text{-Mod}$ such that:

1. For any finite G -sets X and Y , the maps

$$M(X) \oplus M(Y) \begin{array}{c} \xrightarrow{(M_*(\underline{i}_X), M_*(\underline{i}_Y))} \\ \xleftarrow{(M^*(\underline{i}_X), M^*(\underline{i}_Y))} \end{array} M(X \sqcup Y)$$

induced by the canonical inclusions $i_X : X \rightarrow X \sqcup Y$ and $i_Y : Y \rightarrow X \sqcup Y$ are mutual inverse isomorphisms.

2. If

$$\begin{array}{ccc} & X \times_{\underline{a}, \underline{b}} Y & \\ \underline{p} \swarrow & & \searrow \underline{q} \\ X & & Y \\ \underline{a} \searrow & & \swarrow \underline{b} \\ & Z & \end{array}$$

is a weak pullback diagram in $G\text{-}\underline{\mathbf{set}}$, then $M^*(\underline{a})M_*(\underline{b}) = M_*(\underline{p})M^*(\underline{q})$.

A morphism of fused Mackey functors is a natural transformation of bivariant functors.

The category $\mathbf{Mack}_R^f(G)$ can be viewed as a full subcategory of the category $\mathbf{Mack}_R(G)$ of Mackey functors for G over R . In the case $R = \mathbb{Z}$, this category is equivalent to the category of conjugation invariant Mackey functors introduced in [5].

The inclusion functor $\mathbf{Mack}_R^f(G) \hookrightarrow \mathbf{Mack}_R(G)$ has a left adjoint:

4.3. Definition : Let M be a Mackey functor for G over R , in the sense of Lindner, i.e. an R -linear functor $R\mathbf{S}(G) \rightarrow R\text{-}\mathbf{Mod}$. When X is a finite G -set, set

$$M^f(X) = M(X) / \sum_{Z, a, u} \text{Im}(M(\Lambda_{a, \text{Id}_Z}) - M(\Lambda_{u*a, \text{Id}_Z})) ,$$

where the summation runs through triples (Z, a, u) consisting of a finite G -set Z , and morphisms of G -sets $a : Z \rightarrow X$ and $u : Z \rightarrow G^c$, and Λ_{a, Id_Z} denotes the span

$$\begin{array}{ccc} & Z & \\ a \swarrow & & \searrow \text{Id}_Z \\ X & & Z \end{array}$$

of G -sets.

4.4. Proposition : *Let R be a commutative ring, and G be a finite group.*

1. Let M be a Mackey functor for G over R . The correspondence

$$X \mapsto M^f(X)$$

is a fused functor M^f for G over R .

2. The correspondence $\mathcal{F} : M \mapsto M^f$ is a functor from $\mathbf{Mack}_R(G)$ to $\mathbf{Mack}_R^f(G)$, which is left adjoint to the inclusion functor

$$\mathcal{I} : \mathbf{Mack}_R^f(G) \hookrightarrow \mathbf{Mack}_R(G) .$$

Moreover $\mathcal{F} \circ \mathcal{I}$ is isomorphic to the identity functor of $\mathbf{Mack}_R^f(G)$.

Proof : For Assertion 1, to prove that M^f is a Mackey functor, observe that if $\Lambda_{Z,a,b}$ is a span of finite G -sets of the form

$$\begin{array}{ccc} & Z & \\ a \swarrow & & \searrow b \\ X & & Y \end{array}$$

and $u : Z \rightarrow G^c$ is a morphism of G -sets, then

$$\Lambda_{Z,a,b} - \Lambda_{Z,u*a,b} = (\Lambda_{Z,a,\text{Id}_Z} - \Lambda_{Z,u*a,\text{Id}_Z}) \circ \Lambda_{Z,\text{Id}_Z,b} .$$

It follows that the R -module

$$\sum_{Z,a,u} \text{Im}(M(\Lambda_{a,\text{Id}_Z}) - M(\Lambda_{u*a,\text{Id}_Z}))$$

is equal to the sum

$$\sum_{Z,a,b,u} \text{Im}(M(\Lambda_{a,b}) - M(\Lambda_{u*a,b})) .$$

In other words, it is equal to the image by M of the R -submodule $K_R(X, Y)$ of $\text{Hom}_{R\mathbf{S}(G)}(Y, X)$ generated by the morphisms $\Lambda_{a,b} - \Lambda_{u*a,b}$, i.e. to the kernel of the quotient morphism

$$\text{Hom}_{R\mathbf{S}(G)}(Y, X) \rightarrow \text{Hom}_{R\mathbf{S}(G)}(Y, X) .$$

This shows that K_R is an ideal in the category $R\mathbf{S}(G)$. So if M is an R -linear functor $R\mathbf{S}(G) \rightarrow R\mathbf{Mod}$, the correspondence

$$X \mapsto M^f(X) = M(X) / \sum_{f \in K_R(X,Y)} \text{Im} M(f)$$

is an R -linear functor from the quotient category $R\mathbf{S}(G)$ to $R\text{-Mod}$.

Assertion 2 is straightforward: first it is clear that $\mathcal{F} \circ \mathcal{I}$ is isomorphic to the identity functor, since $N^f = N$ when N is a fused Mackey functor. This isomorphism $\mathcal{F} \circ \mathcal{I} \cong \text{Id}_{\text{Mack}_R^f(G)}$ provides the counit of the adjunction. Next for any Mackey functor M , there is a projection morphism $M \rightarrow \mathcal{IF}(M)$, and this yields the unit of the adjunction. \square

4.5. Remark : Assertion 2 shows that $\text{Mack}_R^f(G)$ is a *reflective* subcategory of $\text{Mack}_R^f(G)$ (cf. [7], Chapter IV, Section3).

4.6. Remark : If the Mackey functor M is given in the sense of Dress, then for any finite G -set X

$$M^f(X) = M(X) / \sum_{\substack{a: Z \rightarrow X \\ u: Z \rightarrow G^c}} \text{Im}(M_*(a) - M_*(u * a)) ,$$

where Z is a finite G -set, and a, u are morphisms of G -sets.

4.7. Corollary :

1. If P is a projective Mackey functor, then P^f is projective in the category $\text{Mack}_R^f(G)$.
2. The category $\text{Mack}_R^f(G)$ has enough projective objects. More precisely, if N is a fused Mackey functor, and $\theta : P \rightarrow \mathcal{I}(N)$ is an epimorphism in $\text{Mack}_R(G)$ from a projective Mackey functor P , then $\mathcal{F}(\theta) : P^f \rightarrow N$ is an epimorphism in $\text{Mack}_R^f(G)$.

Proof : Assertion 1 follows from the fact that \mathcal{F} is left adjoint to the exact functor \mathcal{I} . Assertion 2 is then straightforward. \square

5. The fused Mackey algebra

When G is a finite group, set $\Omega_G = \bigsqcup_{H \leq G} G/H$, and let RB_{Ω_G} denote the Dress construction for the Burnside functor RB over the ring R . Recall that RB_{Ω_G} , as a Mackey functor in the sense of Dress, is obtained by precomposition of RB with the endofunctor $X \mapsto X \times \Omega_G$ of G -set.

Also recall (cf. [2] Lemma 7.3.2 and Proposition 4.5.1) that the functor RB_{Ω_G} is a progenerator of the category $\text{Mack}_R(G)$, and that the algebra $\text{End}_{\text{Mack}_R(G)}(B_{\Omega_G}) \cong B(\Omega_G^2)$ is isomorphic to the Mackey algebra $\mu_R(G)$ of G over R , introduced by Thévenaz and Webb ([8]).

It follows from Corollary 4.7 that the functor $(RB_{\Omega_G})^f$ is a progenerator in the category $\mathbf{Mack}_R^f(G)$. Hence this category is equivalent to the category of modules over the algebra $\text{End}_{\mathbf{Mack}_R^f(G)}((RB_{\Omega_G})^f)$.

5.1. Definition : *The fused Mackey algebra of G over R is the algebra*

$$\mu_R^f(G) = \text{End}_{\mathbf{Mack}_R^f(G)}((RB_{\Omega_G})^f) .$$

5.2. Lemma : *Let X be a finite G -set. Then $(RB_X)^f$ is isomorphic to the Yoneda functor $\text{Hom}_{R\mathbf{S}(G)}(X, -)$.*

Proof : Denote by \mathcal{Y}_X the Yoneda functor $\text{Hom}_{R\mathbf{S}(G)}(X, -)$. For any fused Mackey functor N for G over R

$$\begin{aligned} \text{Hom}_{\mathbf{Mack}_R^f(G)}((RB_X)^f, N) &\cong \text{Hom}_{\mathbf{Mack}_R(G)}(RB_X, \mathcal{I}(N)) \\ &\cong \mathcal{I}(N)(X) \cong N(X) \\ &\cong \text{Hom}_{\mathbf{Mack}_R^f(G)}(\mathcal{Y}_X, N) . \end{aligned}$$

The lemma follows, since all these isomorphisms are natural. \square

5.3. Theorem : *The fused Mackey algebra $\mu_R^f(G)$ is isomorphic to the quotient of the algebra $RB(\Omega_G^2) \cong \mu_R(G)$ by the R -module generated by differences of the form*

$$\begin{array}{ccc} & Z & \\ b \swarrow & & \searrow a \\ \Omega_G & & \Omega_G \end{array} \quad - \quad \begin{array}{ccc} & Z & \\ u*b \swarrow & & \searrow a \\ \Omega_G & & \Omega_G , \end{array}$$

where $a, b : Z \rightarrow \Omega_G$ and $u : Z \rightarrow G^c$ are morphisms of G -sets.

Proof : This follows from Lemma 5.2, since the quotient in the theorem is precisely $\text{End}_{R\mathbf{S}(G)}(\Omega_G)$. \square

5.4. Remark : One can deduce from this theorem that the fused Mackey algebra $\mu_R^f(G)$ is always free of finite rank as an R -module, and this rank does not depend on the commutative ring R . More precisely, Thévenaz and Webb have shown ([8] Proposition 3.2) that the Mackey algebra $\mu_R(G)$ has

an R -basis consisting of elements of the form

$$t_K^H c_{g,K} r_{K^g}^L ,$$

where (H, L, g, K) runs through a set of representatives of 4-tuples consisting of two subgroups H and L of G , and element g of G , and a subgroup K of $H \cap {}^g L$, for the equivalence relation \equiv given by

$$(H, L, g, K) \equiv (H', L', g', K') \Leftrightarrow \begin{cases} H = H', L = L', \\ \text{and} \\ \exists h \in H, \exists l \in L, g' = hgl, K' = {}^h K . \end{cases}$$

Similarly, the quotient algebra $\mu_R^f(G)$ of $\mu_R(G)$ has a basis consisting of the images of the elements $t_K^H c_{g,K} r_{K^g}^L$, where (H, L, g, K) runs through a set of representatives of 4-tuples as above, modulo the relation \equiv^f defined by

$$(H, L, g, K) \equiv^f (H', L', g', K') \Leftrightarrow \begin{cases} H = H', L = L', \\ \text{and} \\ \exists h \in H, \exists l \in L, \exists x \in C_G(K), \\ g' = hxgl, K' = {}^h K . \end{cases}$$

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